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Solutions to The Homogeneous Chebyshev's Equation by Means of N-Fractional Calculus Operator

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Abstract

In this article, solutions to homogeneous Chebyshev's equations

$$\varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot \nu^2 = 0, \qquad (\nu \in R,\ z^2 - 1 \neq 0)$$

$$(\ \varphi_\alpha = d^\alpha \varphi / d z^\alpha \ \text{ for } \alpha > 0\ .\ \varphi_0 = \varphi = \varphi(z).\ )$$

are discussed by means of N- fractional calculus operator (NFCO- Method).

By our method the following fractional differintegrated form solutions to the homogeneous Chebyshev's equation are obtained for example.

Group I.

$$\varphi(z) = \left((z^2 - 1)^{-(\nu+1/2)}\right)_{-(1+\nu)} \equiv \varphi_{[1](z,\nu)} \ , \qquad \text{(denote)}$$

$$\varphi(z) = \left((z^2 - 1)^{\nu-1/2}\right)_{\nu-1} \equiv \varphi_{[2](z,\nu)} \ .$$

Group II.

$$\varphi(z) = (z^2 - 1)^{1/2}\left((z^2 - 1)^{-(\nu+1/2)}\right)_{-\nu} \equiv \varphi_{[3](z,\nu)} \ ,$$

$$\varphi(z) = (z^2 - 1)^{1/2}\left((z^2 - 1)^{\nu-1/2}\right)_{\nu} \equiv \varphi_{[4](z,\nu)} \ .$$

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin \mathbf{Z}^-), \qquad (1)$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in \mathbf{Z}^+), \qquad (2)$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $\; z \in \mathbf{C}$, $\; \nu \in \mathbf{R}$, $\; \Gamma$; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

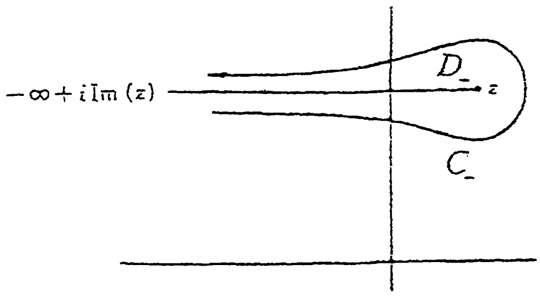

Fig. 1.

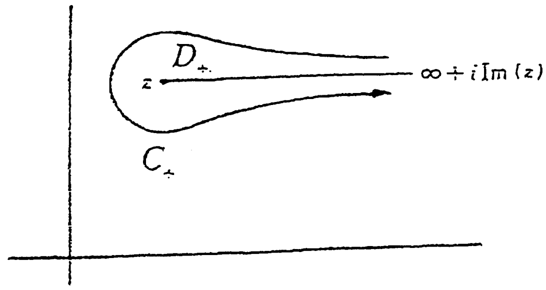

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu = n(\in \mathbf{Z}^+)$ and is reduced to the famous Cauchy's integral for $\nu = 0$. That is, (1) is an extension of Cauchy integral and of Goursat's one, conversely Cauchy and Goursat's ones are the special cases of (1).

(II) On the fractional calculus operator N^{v} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{v} be*

$$N^{v}=\left(\frac{\Gamma(v+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{v+1}}\right)\qquad(v\notin \boldsymbol{Z}),\quad [\,\textit{Refer to}\,(1)\,]\tag{3}$$

with
$$N^{-m}=\lim_{v\to -m}N^{v}\qquad(m\in \boldsymbol{Z}^{+}),\tag{4}$$

and define the binary operation $\circ$ as

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in \boldsymbol{R}),\tag{5}$$

then the set

$$\left\{N^{v}\right\}=\left\{N^{v}\mid v\in \boldsymbol{R}\right\}\tag{6}$$

is an Abelian product group (having continuous index v) which has the inverse transform operator $(N^{v})^{-1}=N^{-v}$ to the fractional calculus operator N^{v} , for the function f such that $f\in F=\left\{f;\ 0\neq|f_{v}|<\infty, v\in \boldsymbol{R}\right\}$, where $f=f(z)$ and $z\in \boldsymbol{C}$. (vis. $-\infty<v<\infty$).

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{v}\}$ " *is an " Action product group which has continuous index v " for the set of F . (F.O.G. ;* Fractional calculus operator group) [3]

Theorem C. *Let*

$$S:=\{\pm N^{v}\}\cup\{0\}=\{N^{v}\}\cup\{-N^{v}\}\cup\{0\}\quad(v\in \boldsymbol{R}).\tag{7}$$

Then the set S is a commutative ring for the function $f\in F$, when the identity

$$N^{\alpha}+N^{\beta}=N^{\gamma}\qquad(N^{\alpha},N^{\beta},N^{\gamma}\in S),\tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^{b})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right),$$

(ii) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad(|\Gamma(\alpha)|<\infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad(|\Gamma(\alpha)|<\infty),$$

where $z-c\neq 0$ for (i) and $z-c\neq 0,1$ for (ii), (iii) ,

(iv) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\left(\begin{array}{l}u=u(z),\\ v=v(z)\end{array}\right).$$

§ 1. Preliminary

(I) The theorem below is reported by the author already (cf. J.F C, Vol. 27, May (2005), 83 - 88.). [31]

Theorem D. *Let*

$$P = P(\alpha,\beta,\gamma) := \frac{\sin\pi\alpha\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\alpha)} \quad \left(|P(\alpha,\beta,\gamma)| = M < \infty\right) \qquad (1)$$

and

$$Q = Q(\alpha,\beta,\gamma) := P(\beta,\alpha,\gamma), \qquad (|P(\beta,\alpha,\gamma)| = M < \infty) \qquad (2)$$

When $\alpha,\beta,\gamma \notin Z_0^+$ *, we have* ;

(i) $$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma = e^{-i\pi\gamma}P(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (3)$$

$$(\mathrm{Re}(\alpha+\beta+1) > 0,\ (1+\alpha-\gamma)\notin Z_0^-),$$

(ii) $$((z-c)^\beta\cdot(z-c)^\alpha)_\gamma = e^{-i\pi\gamma}Q(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (4)$$

$$(\mathrm{Re}(\alpha+\beta+1) > 0,\ (1+\beta-\gamma)\notin Z_0^-)$$

(iii) $$((z-c)^{\alpha+\beta})_\gamma = e^{-i\pi\gamma}\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (5)$$

where

$$z-c\neq 0, \quad \left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right| < \infty .$$

Then the inequalities below are established from this theorem.

Corollary 1. *We have the inequalities*

(i) $$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma \neq ((z-c)^\beta\cdot(z-c)^\alpha)_\gamma, \qquad (6)$$

and

(ii) $$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma \neq ((z-c)^{\alpha+\beta})_\gamma, \qquad (7)$$

where

$$\alpha,\beta,\gamma\notin Z_0^+, \quad \alpha\neq\beta, \quad z-c\neq 0.$$

Corollary 2.

(i) *When* $\alpha, \beta, \gamma \notin \mathbf{Z}_0^+$, *and*

$$P(\alpha, \beta, \gamma) = Q(\beta, \alpha, \gamma) = 1 , \qquad (8)$$

we have

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma} = ((z-c)^{\alpha+\beta})_{\gamma} , \qquad (9)$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, \quad (1+\alpha-\gamma) \notin Z_0^- , \, (1+\beta-\gamma) \notin Z_0^-).$$

(i i) *When* $\gamma = m \in \mathbf{Z}_0^+$, *we have* ;

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_m = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_m = ((z-c)^{\alpha+\beta})_m . \qquad (10)$$

(I I) The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006),pp.35 - 44.) . [32]

Theorem E . *We have*

(i)

$$\left(((z-b)^{\beta} - c)^{\alpha}\right)_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \, \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}}\right)^k \qquad (11)$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right)$$

and

(i i)

$$\left(((z-b)^{\beta} - c)^{\alpha}\right)_n = (-1)^n (z-b)^{\alpha\beta-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^{\beta}}\right)^k \qquad (n \in \mathbf{Z}_0^+) \qquad (12)$$

where

$$\left| \frac{c}{(z-b)^{\beta}} \right| < 1 ,$$

and

$$[\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \text{ with } [\lambda]_0 = 1 ,$$

(Notation of Pochhammer).

§ 2. Solutions to The Homogeneous Chebyshev's Equations by Means of N- Fractional Calculus Operator

Theorem 1. *Let* $\varphi = \varphi(z) \in F$, *then the homogeneous Chebyshev's equation*

$$L[\varphi ; z ; \nu] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot \nu^2 = 0 \quad (\nu \in R,\ z^2 - 1 \neq 0) \qquad (1)$$

$$(\ \varphi_\alpha = d^\alpha \varphi / d z^\alpha \ \text{ for } \alpha > 0 .\ \ \varphi_0 = \varphi = \varphi(z).\ )$$

has particular solutions of the forms (fractional differintegrated fotm);

Group i.

(i) $$\varphi(z) = \left((z^2 - 1)^{-(\nu+1/2)}\right)_{-(1+\nu)} \equiv \varphi_{[1](z,\nu)} , \quad \text{(denote)} \qquad (2)$$

(ii) $$\varphi(z) = \left((z^2 - 1)^{\nu-1/2}\right)_{\nu-1} \equiv \varphi_{[2](z,\nu)} , \qquad (3)$$

Group I I.

(i) $$\varphi(z) = (z^2 - 1)^{1/2}\left((z^2 - 1)^{-(\nu+1/2)}\right)_{-\nu} \equiv \varphi_{[3](z,\nu)} , \qquad (4)$$

(ii) $$\varphi(z) = (z^2 - 1)^{1/2}\left((z^2 - 1)^{\nu-1/2}\right)_{\nu} \equiv \varphi_{[4](z,\nu)} , \qquad (5)$$

Group I I I.

(i) $$\varphi(z) = (z-1)^{1/2}\left((z-1)^{-(\nu+1)} \cdot (z+1)^{-\nu}\right)_{-(\nu+1/2)} \equiv \varphi_{[5](z,\nu)} , \qquad (6)$$

(ii) $$\varphi(z) = (z-1)^{1/2}\left((z+1)^{-\nu} \cdot (z-1)^{-(\nu+1)}\right)_{-(\nu+1/2)} \equiv \varphi_{[6](z,\nu)} \qquad (7)$$

(iii) $$\varphi(z) = (z-1)^{1/2}\left((z-1)^{\nu-1} \cdot (z+1)^{\nu}\right)_{\nu-1/2} \equiv \varphi_{[7](z,\nu)} . \qquad (8)$$

(iv) $$\varphi(z) = (z-1)^{1/2}\left((z+1)^{\nu} \cdot (z-1)^{\nu-1}\right)_{\nu-1/2} \equiv \varphi_{[8](z,\nu)} , \qquad (9)$$

Group I V.

(i) $$\varphi(z) = (z+1)^{1/2}\left((z-1)^{-\nu} \cdot (z+1)^{-(\nu+1)}\right)_{-(\nu+1/2)} \equiv \varphi_{[9](z,\nu)} , \qquad (10)$$

(ii) $$\varphi(z) = (z+1)^{1/2}\left((z+1)^{-(\nu+1)} \cdot (z-1)^{-\nu}\right)_{-(\nu+1/2)} \equiv \varphi_{[10](z,\nu)} , \qquad (11)$$

(iii) $$\varphi(z) = (z+1)^{1/2}\left((z-1)^{\nu} \cdot (z+1)^{\nu-1}\right)_{\nu-1/2} \equiv \varphi_{[11](z,\nu)} , \qquad (12)$$

(iv) $$\varphi(z) = (z+1)^{1/2}\left((z+1)^{\nu-1} \cdot (z-1)^{\nu}\right)_{\nu-1/2} \equiv \varphi_{[12](z,\nu)} . \qquad (13)$$

Proof of Group I ;

Operate N-fractional calculus operator N^{α} to the both sides of (1), we have then

$$(\varphi_2 \cdot (z^2 - 1))_{\alpha} + (\varphi_1 \cdot z)_{\alpha} - (\varphi \cdot \nu^2)_{\alpha} = 0 \quad (\alpha \notin \mathbf{Z}^-). \tag{14}$$

Now we have

$$(\varphi_2 \cdot (z^2 - 1))_{\alpha} = \sum_{k=0}^{2} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} (\varphi_2)_{\alpha-k} (z^2-1)_k \tag{15}$$

$$= \varphi_{2+\alpha} \cdot (z^2 - 1) + \varphi_{1+\alpha} \cdot z2\alpha + \varphi_{\alpha} \cdot \alpha(\alpha - 1), \tag{16}$$

$$(\varphi_1 \cdot z)_{\alpha} = \sum_{k=0}^{1} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} (\varphi_1)_{\alpha-k} (z)_k \tag{17}$$

$$= \varphi_{1+\alpha} \cdot z + \varphi_{\alpha} \cdot \alpha \ , \tag{18}$$

and

$$(\varphi \cdot \nu^2)_{\alpha} = \varphi_{\alpha} \cdot \nu^2, \tag{19}$$

by Lemma (i v), respectively.

Therefore, we obtain

$$\varphi_{2+\alpha} \cdot (z^2 - 1) + \varphi_{1+\alpha} \cdot z(2\alpha + 1) + \varphi_{\alpha} \cdot (\alpha^2 - \nu^2) = 0, \tag{20}$$

from (14),applying (16), (18) and (19).

Choose α such that

$$\alpha^2 - \nu^2 = 0,$$

yields

$$\alpha = \nu, \quad -\nu \ . \tag{21}$$

(I) Case $\alpha = \nu$;

In this case we obtain

$$\varphi_{2+\nu} \cdot (z^2 - 1) + \varphi_{1+\nu} \cdot z(2\nu + 1) = 0, \tag{22}$$

from (20), setting $\alpha = \nu$.

Let

$$\varphi_{1+\nu} = \psi = \psi(z) \qquad (\varphi = \psi_{-(1+\nu)}), \tag{23}$$

we have then

$$\psi_1 \cdot (z^2 - 1) + \psi \cdot z(2\nu + 1) = 0 \tag{24}$$

from (22). A particular solution to this equation (24) (variables separable form) is given by

$$\psi = (z^2 - 1)^{-(\nu+1/2)} \quad . \tag{25}$$

Therefore, we obtain

$$\varphi(z) = \left((z^2 - 1)^{-(\nu+1/2)}\right)_{-(1+\nu)} \equiv \varphi_{[1](z,\nu)} \quad , \tag{2}$$

from (23) and (25).

Inversely, we obtain

$$\varphi_{2+\nu} = \left((z^2 - 1)^{-(\nu+1/2)}\right)_1 \quad , \tag{26}$$

and

$$\varphi_{1+\nu} = (z^2 - 1)^{-(\nu+1/2)} \quad , \tag{27}$$

from (2), respectively.

Hence we obtain

$$\text{LHS of } (22) = ((z^2 - 1)^{-(\nu+1/2)})_1 \cdot (z^2 - 1) + (z^2 - 1)^{-(\nu+1/2)} \cdot z(2\nu + 1) = 0 \ , \tag{28}$$

applying (26) and (27).

Therefore, the functions shown by (2) satisfy the equation (1), clearly.

(I I) Case $\alpha = -\nu$;

In the same way as the proof of **(I)** (setting $-\nu$ instead of ν in (2)), we obtain

$$\varphi(z) = \left((z^2 - 1)^{\nu-1/2}\right)_{\nu-1} \equiv \varphi_{[2](z,\nu)} \quad , \tag{3}$$

and clearly

$$\varphi_{[2](z,\nu)} = \varphi_{[1](z,-\nu)} \ .$$

Proof of Group I I ;

Set

$$\varphi(z) = (z^2 - 1)^{\lambda}\phi \ , \qquad \phi = \phi(z) \ , \tag{29}$$

we have then

$$\varphi_1 = \lambda(z^2 - 1)^{\lambda-1} 2\, z\phi + (z^2 - 1)^{\lambda}\phi_1 \ , \tag{30}$$

and

$$\varphi_2 = \lambda(\lambda - 1)(z^2 - 1)^{\lambda-2}(2z)^2\phi + \lambda(z^2 - 1)^{\lambda-1} 2\phi + \lambda(z^2 - 1)^{\lambda-1} 4\, z\phi_1 + (z^2 - 1)^{\lambda}\phi_2 \tag{31}$$

from (29), respectively.

Substituting (29), (30) and (31) into (1), yields

$$\phi_2 \cdot (z^2 - 1)^{\lambda+1} + \phi_1 \cdot (z^2 - 1)^{\lambda}\{z(4\lambda + 1)\} + \phi \cdot (z^2 - 1)^{\lambda}\left\{(4\lambda^2 - \nu^2) + \frac{2\lambda(2\lambda - 1)}{z^2 - 1}\right\} = 0. \tag{32}$$

Choose λ such that

$$\lambda(2\lambda-1)=0,$$

yields

$$\lambda=0,\quad 1/2\ . \qquad (33)$$

When $\lambda=0$, (32) is reduced to (1). We have then the same particular solutions as (2) and (3).

When $\lambda=1/2$, we have

$$\phi_2\cdot(z^2-1)+\phi_1\cdot 3z+\phi\cdot(1-\nu^2)=0 \qquad (34)$$

from (32).

Operate N^{α} to the both sides of (34), then we obtain

$$\phi_{2+\alpha}\cdot(z^2-1)+\phi_{1+\alpha}\cdot z(2\alpha+3)+\phi_{\alpha}\cdot(\alpha^2+2\alpha+1-\nu^2)=0 \quad (\alpha\notin \mathbf{Z}^-). \qquad (35)$$

Choose α such that

$$(\alpha+1)^2-\nu^2=0,$$

gives

$$\alpha=\nu-1,\quad -\nu-1\ . \qquad (36)$$

(I) Case $\alpha=\nu-1$;

Setting

$$\phi_\nu=V=V(z),\quad (\phi=V_{-\nu}), \qquad (37)$$

we have

$$V_1\cdot(z^2-1)+V\cdot z(2\nu+1)=0 \qquad (38)$$

from (35).

A particular solution to this equation is given by

$$V=(z^2-1)^{-(\nu+1/2)}\ . \qquad (39)$$

We have then

$$\phi=V_{-\nu}=\left((z^2-1)^{-(\nu+1/2)}\right)_{-\nu} \qquad (40)$$

from (37) and (39), hence we obtain

$$\varphi=(z^2-1)^{1/2}\left((z^2-1)^{-(\nu+1/2)}\right)_{-\nu}\equiv\varphi_{[3](z,\nu)} \qquad (4)$$

from (29) and (40), for $\lambda=1/2$.

(II) Case $\alpha=-\nu-1$;

In the same way as the proof of **(I)** (setting $-\nu$ instead of ν in (4)), we obtain

$$\varphi(z)=(z^2-1)^{1/2}\left((z^2-1)^{\nu-1/2}\right)_\nu\equiv\varphi_{[4](z,\nu)}\ , \qquad (5)$$

and clearly

$$\varphi_{[4](z,\nu)}=\varphi_{[3](z,-\nu)}\ .$$

Proof of Group III ;

Setting

$$\varphi(z) = (z-1)^{\lambda}\phi\,, \qquad \phi = \phi(z)\ , \tag{41}$$

we obtaun

$$\phi_2\cdot(z^2-1)+\phi_1\cdot\{z(2\lambda+1)+2\lambda\}+\phi\cdot\left\{\lambda^2-\nu^2+\frac{\lambda(2\lambda-1)}{z-1}\right\}=0 \tag{42}$$

from (1), applying (41).

Choose λ such that

$$\lambda(2\lambda-1)=0,$$

yields

$$\lambda = 0\,,\quad 1/2\ . \tag{43}$$

When $\lambda = 0$, (42) is reduced to (1). We have then the same particular solutions as (2) and (3).

When $\lambda = 1/2$, we have

$$\phi_2\cdot(z^2-1)+\phi_1\cdot(2z+1)+\phi\cdot(\tfrac{1}{4}-\nu^2)=0 \tag{44}$$

from (42).

Operate N^{α} to the both sides of (44), then we obtain

$$\phi_{2+\alpha}\cdot(z^2-1)+\phi_{1+\alpha}\cdot\{z(2\alpha+2)+1\}+\phi_{\alpha}\cdot(\alpha^2+\alpha+\tfrac{1}{4}-\nu^2)=0 \quad (\alpha\notin \mathbf{Z}^-). \tag{45}$$

Choose α such that

$$(\alpha+\tfrac{1}{2})^2-\nu^2=0,$$

gives

$$\alpha=\nu-\tfrac{1}{2},\quad -\nu-\tfrac{1}{2}\ . \tag{46}$$

(I) Case $\alpha=\nu-\frac{1}{2}$;

Setting

$$\phi_{\nu+1/2}=V=V(z)\,,\quad (\phi=V_{-(\nu+1/2)})\,, \tag{47}$$

we have

$$V_1+V\cdot\frac{z(1+2\nu)+1}{z^2-1}=0 \tag{48}$$

from (45).

A particular solution to this equation is given by

$$V=(z-1)^{-(\nu+1)}(z+1)^{-\nu}\ . \tag{49}$$

Thus we obtain a particular solution

$$\varphi=(z-1)^{1/2}\left((z-1)^{-(\nu+1)}\cdot(z+1)^{-\nu}\right)_{-(\nu+1/2)}\equiv\varphi_{[5](z,\nu)} \tag{6}$$

from (41) , applying (49) and (47), for $\lambda = 1/2$.

Changing the order

$(z-1)^{-(\nu+1)}$ and $(z+1)^{-\nu}$ in parenthesis $(\quad \cdot \quad)_{-(\nu+1/2)}$ in (6)

we obtain

$$\varphi = (z-1)^{1/2}\left((z+1)^{-\nu}\cdot(z-1)^{-(\nu+1)}\right)_{-(\nu+1/2)} \equiv \varphi_{[6](z,\nu)} \tag{7}$$

where

$$\varphi_{[5](z,\nu)} \neq \varphi_{[6](z,\nu)} \quad (\text{for } -(\nu+1/2) \notin \boldsymbol{Z}_0^+) \ . \tag{50}$$

(I I) Case $\alpha = -\nu - \frac{1}{2}$;

In the same way as the proof of **(I)** (setting $-\nu$ instead of ν in (6) and (7)), we obtain

$$\varphi = (z-1)^{1/2}\left((z-1)^{\nu-1}\cdot(z+1)^{\nu}\right)_{\nu-1/2} \equiv \varphi_{[7](z,\nu)} \ , \tag{8}$$

$$\varphi = (z-1)^{1/2}\left((z+1)^{\nu}\cdot(z-1)^{\nu-1}\right)_{\nu-1/2} \equiv \varphi_{[8](z,\nu)} \tag{9}$$

and

$$\varphi_{[7](z,\nu)} \neq \varphi_{[8](z,\nu)} \quad (\text{for } (\nu-1/2) \notin \boldsymbol{Z}_0^+) \ . \tag{51}$$

respectively.

Proof of Group I V ;

Set

$$\varphi(z) = (z+1)^{\lambda}\phi \ , \qquad \phi = \phi(z) \ , \tag{52}$$

we have then

$$\phi_2\cdot(z^2-1) + \phi_1\cdot\{z(2\lambda+1) - 2\lambda\} + \phi\cdot\left\{\lambda^2 - \nu^2 - \frac{\lambda(2\lambda-1)}{z+1}\right\} = 0 \tag{53}$$

from (1), applying (52).

Therefore, in the same way as the proof of **Group I I I**, we can obtain the particular solutions (10) ~ (13).

That is, choosing λ such that

$$\lambda(2\lambda-1) = 0,$$

yields

$$\lambda = 0 \ , \quad 1/2 \ . \tag{54}$$

When $\lambda = 0$, (53) is reduced to (1). We have then the same particular solutions as (2) and (3).

When $\lambda = 1/2$, we have

$$\phi_2\cdot(z^2-1) + \phi_1\cdot(2z-1) + \phi\cdot\left(\tfrac{1}{4} - \nu^2\right) = 0 \tag{55}$$

from (53).

Operate N^{α} to the both sides of (55), then we obtain

$$\phi_{2+\alpha}\cdot(z^2-1)+\phi_{1+\alpha}\cdot\{z(2\alpha+2)-1\}+\phi_{\alpha}\cdot(\alpha^2+\alpha+\tfrac{1}{4}-\nu^2)=0 \quad (\alpha\notin \boldsymbol{Z}^-). \tag{56}$$

Choose α such that

$$(\alpha+\tfrac{1}{2})^2-\nu^2=0,$$

gives

$$\alpha=\nu-\tfrac{1}{2}, \quad -\nu-\tfrac{1}{2}\ . \tag{57}$$

(I) Case $\alpha=\nu-\frac{1}{2}$;

Setting

$$\phi_{\nu+1/2}=V=V(z), \quad (\phi=V_{-(\nu+1/2)}), \tag{58}$$

we have

$$V_1+V\cdot\frac{z(1+2\nu)-1}{z^2-1}=0 \tag{59}$$

from (56).

A particular solution to this equation is given by

$$V=(z-1)^{-\nu}(z+1)^{-(\nu+1)}\ . \tag{60}$$

Thus we obtain a particular solution

$$\varphi=(z+1)^{1/2}\left((z-1)^{-\nu}\cdot(z+1)^{-(\nu+1)}\right)_{-(\nu+1/2)}\equiv\varphi_{[9](z,\nu)} \tag{10}$$

from (52) , applying (60) and (58), for $\lambda=1/2$.

Next changing the order

$(z-1)^{-\nu}$ and $(z+1)^{-(\nu+1)}$ in parenthesis $(\quad\cdot\quad)_{-(\nu+1/2)}$ in (10)

we obtain

$$\varphi=(z+1)^{1/2}\left((z+1)^{-(\nu+1)}\cdot(z-1)^{-\nu}\right)_{-(\nu+1/2)}\equiv\varphi_{[10](z,\nu)} \tag{11}$$

where

$$\varphi_{[9](z,\nu)}\neq\varphi_{[10](z,\nu)} \quad (\text{for } -(\nu+1/2)\notin\boldsymbol{Z}_0^+)\ . \tag{61}$$

(II) Case $\alpha=-\nu-\frac{1}{2}$;

In the same way as the proof of **(I)** (setting $-\nu$ instead of ν in (10) and (11)), we obtain

$$\varphi=(z+1)^{1/2}\left((z-1)^{\nu}\cdot(z+1)^{\nu-1}\right)_{\nu-1/2}\equiv\varphi_{[11](z,\nu)}\ , \tag{12}$$

$$\varphi=(z+1)^{1/2}\left((z+1)^{\nu-1}\cdot(z-1)^{\nu}\right)_{\nu-1/2}\equiv\varphi_{[12](z,\nu)} \tag{13}$$

and

$$\varphi_{[11](z,\nu)}\neq\varphi_{[12](z,\nu)} \quad (\text{for } (\nu-1/2)\notin\boldsymbol{Z}_0^+)\ . \tag{62}$$

respectively.

§ 3. Familiar Forms of The Solutions obtained in § 2

Theorem 2. We have the following (more familiar form) presentations for the solutions to homogeneous Chebyshev's equation.

Group I .

(i) $$\varphi_{[1](z,\nu)} = -e^{i\pi\nu}\frac{\sqrt{\pi}}{2^{2\nu}\nu\Gamma(\nu+1/2)}z^{-\nu}\,{}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\nu+1;\frac{1}{z^2}\right), \tag{1}$$

$$\left((\nu+1)\notin \mathbf{Z}_0^-,\ \left|\frac{1}{z^2}\right|<1\right).$$

(ii) $$\varphi_{[2](z,\nu)} = e^{-i\pi\nu}\frac{\sqrt{\pi}}{2^{-2\nu}\nu\Gamma(1/2-\nu)}z^{\nu}\,{}_2F_1\left(-\frac{\nu}{2},\frac{1-\nu}{2};1-\nu;\frac{1}{z^2}\right), \tag{2}$$

$$\left((1-\nu)\notin \mathbf{Z}_0^-,\ \left|\frac{1}{z^2}\right|<1\right).$$

Group I I .

(i) $$\varphi_{[3](z,\nu)} = e^{i\pi\nu}\frac{\sqrt{\pi}}{2^{2\nu}\Gamma(\nu+1/2)}(z^2-1)^{1/2}z^{-(\nu+1)}\,{}_2F_1\left(\frac{\nu}{2}+1,\frac{\nu+1}{2};\nu+1;\frac{1}{z^2}\right), \tag{3}$$

$$\left((\nu+1)\notin \mathbf{Z}_0^-,\ \left|\frac{1}{z^2}\right|<1\right).$$

(ii) $$\varphi_{[4](z,\nu)} = e^{-i\pi\nu}\frac{\sqrt{\pi}}{2^{-2\nu}\Gamma(1/2-\nu)}(z^2-1)^{1/2}z^{\nu-1}\,{}_2F_1\left(1-\frac{\nu}{2},\frac{1-\nu}{2};1-\nu;\frac{1}{z^2}\right), \tag{4}$$

$$\left((1-\nu)\notin \mathbf{Z}_0^-,\ \left|\frac{1}{z^2}\right|<1\right).$$

Group I I I .

(i) $$\varphi_{[5](z,\nu)} = ie^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z+1)^{-\nu}\,{}_2F_1\left(\frac{1}{2}+\nu,\nu;\frac{1}{2};\frac{z-1}{z+1}\right), \tag{5}$$

$$\left(\left|\frac{z-1}{z+1}\right|<1\right).$$

(ii) $$\varphi_{[6](z,\nu)} = -ie^{i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(\nu)}(z-1)^{-\nu}\left(\frac{z+1}{z-1}\right)^{1/2}\,{}_2F_1\left(\frac{1}{2}+\nu,1+\nu;\frac{3}{2};\frac{z+1}{z-1}\right), \tag{6}$$

$$\left(\left|\frac{z+1}{z-1}\right|<1\right).$$

(iii) $$\varphi_{[7](z,\nu)} = ie^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1-\nu)}(z+1)^{\nu}\,{}_2F_1\left(\frac{1}{2}-\nu,-\nu;\frac{1}{2};\frac{z-1}{z+1}\right), \tag{7}$$

$$\left(\left|\frac{z-1}{z+1}\right|<1\right).$$

(iv) $$\varphi_{[8](z,\nu)} = -ie^{-i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(-\nu)}(z-1)^{\nu}\left(\frac{z+1}{z-1}\right)^{1/2}\,{}_2F_1\left(\frac{1}{2}-\nu,1-\nu;\frac{3}{2};\frac{z+1}{z-1}\right), \tag{8}$$

$$\left(\left|\frac{z+1}{z-1}\right|<1\right).$$

Group I V .

(i) $$\varphi_{[9](z,\nu)} = -ie^{i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(\nu)}(z+1)^{-\nu}\left(\tfrac{z-1}{z+1}\right)^{1/2} {}_2F_1\left(\tfrac{1}{2}+\nu\,,1+\nu\,;\tfrac{3}{2}\,;\tfrac{z-1}{z+1}\right), \quad (9)$$

$$\left(\left|\tfrac{z-1}{z+1}\right|<1\right).$$

(ii) $$\varphi_{[10](z,\nu)} = ie^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z-1)^{-\nu}\ {}_2F_1\left(\tfrac{1}{2}+\nu\,,\nu\,;\tfrac{1}{2}\,;\tfrac{z+1}{z-1}\right), \quad (10)$$

$$\left(\left|\tfrac{z+1}{z-1}\right|<1\right).$$

(iii) $$\varphi_{[11](z,\nu)} = -ie^{-i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(-\nu)}(z+1)^{\nu}\left(\tfrac{z-1}{z+1}\right)^{1/2} {}_2F_1\left(\tfrac{1}{2}-\nu\,,1-\nu\,;\tfrac{3}{2}\,;\tfrac{z-1}{z+1}\right), \quad (11)$$

$$\left(\left|\tfrac{z-1}{z+1}\right|<1\right).$$

(iv) $$\varphi_{[12](z,\nu)} = ie^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1-\nu)}(z-1)^{\nu}\ {}_2F_1\left(\tfrac{1}{2}-\nu\,,-\nu\,;\tfrac{1}{2}\,;\tfrac{z+1}{z-1}\right), \quad (12)$$

$$\left(\left|\tfrac{z+1}{z-1}\right|<1\right)$$

Where ${}_2F_1(\cdots\cdots)$ is the usual Gauss hypergeometric function.

Proof of Group I ;

We have the below using Theorem E (i) in § 1 .

(i) $$\varphi_{[1](z,\nu)} = \left((z^2-1)^{-(\nu+1/2)}\right)_{-(1+\nu)} \quad (13)$$

$$= -e^{i\pi\nu}z^{-\nu}\sum_{k=0}^{\infty}\frac{[\nu+\frac{1}{2}]_k\Gamma(2k+\nu)}{k!\,\Gamma(2k+2\nu+1)}\left(\frac{1}{z^2}\right)^k \quad (|1/z^2|<1) \quad (14)$$

$$= -e^{i\pi\nu}2^{-(\nu+1)}\frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu+1}{2})}{\Gamma(\nu+\frac{1}{2})\Gamma(\nu+1)}z^{-\nu}\sum_{k=0}^{\infty}\frac{\left[\frac{\nu}{2}\right]_k\left[\frac{\nu+1}{2}\right]_k}{k!\,[\nu+1]_k}\left(\frac{1}{z^2}\right)^k \quad (15)$$

$$= -e^{i\pi\nu}\frac{\sqrt{\pi}}{2^{2\nu}\nu\Gamma(\nu+\frac{1}{2})}z^{-\nu}\ {}_2F_1\left(\tfrac{\nu}{2}\,,\tfrac{\nu+1}{2}\ ;\nu+1\ ;\tfrac{1}{z^2}\right) \quad (1)$$

$$\left((\nu+1)\notin \boldsymbol{Z}_0^-\right).$$

since we have

$$\frac{\sqrt{\pi}}{2^{\nu-1}}\Gamma(\nu) = \Gamma(\tfrac{\nu}{2})\Gamma(\tfrac{\nu+1}{2}) \quad (16)$$

by Legendre's identity

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \ . \quad (17)$$

We have then

$$\Gamma(2k+\nu)=\Gamma(2(k+\tfrac{\nu}{2}))=\frac{2^{2k+\nu-1}}{\sqrt{\pi}}\Gamma(k+\tfrac{\nu}{2})\Gamma(k+\tfrac{\nu}{2}+\tfrac{1}{2}) \quad , \tag{18}$$

$$\Gamma(2k+2\nu+1)=\Gamma(2(k+\nu+\tfrac{1}{2}))=\frac{2^{2k+2\nu}}{\sqrt{\pi}}\Gamma(k+\nu+\tfrac{1}{2})\Gamma(k+\nu+\tfrac{1}{2}+\tfrac{1}{2}) \quad , \tag{19}$$

hence

$$\frac{\Gamma(2k+\nu)}{\Gamma(2k+2\nu+1)}=2^{-(\nu+1)}\frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu+1}{2})[\frac{\nu}{2}]_k[\frac{\nu+1}{2}]_k}{\Gamma(\frac{2\nu+1}{2})\Gamma(\nu+1)[\frac{2\nu+1}{2}]_k[\nu+1]_k} \quad . \tag{20}$$

Next we have

(i i)
$$\varphi_{[2](z,\nu)}=\left((z^2-1)^{\nu-1/2}\right)_{\nu-1}=\varphi_{[1](z,-\nu)} \tag{21}$$

hence setting $-\nu$ instead of ν in (1) , we obtain (2) clearly.

Proof of Group I I ;

We have the below using Theorem E (i) in **§ 1** .

(i)
$$\varphi_{[3](z,\nu)}=(z^2-1)^{1/2}\left((z^2-1)^{-(\nu+1/2)}\right)_{-\nu} \tag{22}$$

$$=e^{i\pi\nu}(z^2-1)^{1/2}z^{-(\nu+1)}\sum_{k=0}^{\infty}\frac{[\nu+\frac{1}{2}]_k\Gamma(2k+\nu+1)}{k\,!\,\Gamma(2k+2\nu+1)}\left(\frac{1}{z^2}\right)^k \tag{23}$$

$$\left(\left|\frac{\Gamma(2k+\nu+1)}{\Gamma(2k+2\nu+1)}\right|<\infty .\quad \left|\frac{1}{z^2}\right|<1\right)$$

$$=e^{i\pi\nu}\frac{\sqrt{\pi}}{2^{2\nu}\Gamma(\nu+\frac{1}{2})}(z^2-1)^{1/2}z^{-(\nu+1)}\ {}_2F_1(\tfrac{\nu}{2}+1\ ,\tfrac{\nu+1}{2}\ ;\nu+1\ ;\tfrac{1}{z^2}) \tag{3}$$

$$\left((\nu+1)\notin \boldsymbol{Z}_0^-\right) .$$

since we have

$$\Gamma(2k+\nu+1)=(2k+\nu)\Gamma(2(k+\tfrac{\nu}{2})) \quad , \tag{24}$$

$$=(2k+\nu)\frac{2^{2k+\nu-1}}{\sqrt{\pi}}\Gamma(k+\tfrac{\nu}{2})\Gamma(k+\tfrac{\nu}{2}+\tfrac{1}{2}) \ , \tag{25}$$

$$=\frac{2^{2k+\nu}}{\sqrt{\pi}}\Gamma(k+\tfrac{\nu}{2}+1)\Gamma(k+\tfrac{\nu}{2}+\tfrac{1}{2}) \ , \tag{26}$$

and (19), and

$$\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}=[\lambda]_k \tag{27}$$

hence

$$\Gamma(k+\tfrac{\nu}{2}+1)=\Gamma(\tfrac{\nu}{2}+1)[\tfrac{\nu}{2}+1]_k \ . \tag{28}$$

$$\Gamma(k+\tfrac{\nu+1}{2}) = \Gamma(\tfrac{\nu+1}{2})[\tfrac{\nu}{2}+\tfrac{1}{2}]_k \ . \tag{29}$$

$$\Gamma(k+\nu+1) = \Gamma(\nu+1)[\nu+1]_k \ . \tag{30}$$

Next we have

(ii) $$\varphi_{[4](z,\nu)} = (z^2-1)^{1/2}\left((z^2-1)^{\nu-1/2}\right)_\nu = \varphi_{[3](z,-\nu)} \ . \tag{31}$$

Hence setting $-\nu$ instead of ν in (3), we obtain (4) clearly.

Proof of Group III ;

We have

(i) $$\varphi_{[5](z,\nu)} = (z-1)^{1/2}\left((z-1)^{-(\nu+1)}\cdot(z+1)^{-\nu}\right)_{-(\nu+1/2)} \tag{32}$$

$$= (z-1)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(1-\nu-\frac{1}{2})}{k!\,\Gamma(1-\nu-\frac{1}{2}-k)}((z-1)^{-(\nu+1)})_{-(\nu+1/2)-k}((z+1)^{-\nu})_k \tag{33}$$

(by Lemma (iv))

$$= i\,e^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z-1)^{-\nu}\sum_{k=0}^{\infty}\frac{[\frac{1}{2}+\nu]_k\,[\nu]_k}{k!\,[\frac{1}{2}]_k}\left(\frac{z-1}{z+1}\right)^k \quad \text{(by Lemma (i))} \tag{34}$$

$$= i\,e^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z+1)^{-\nu}\ {}_2F_1(\tfrac{1}{2}+\nu,\nu;\tfrac{1}{2};\tfrac{z-1}{z+1}) \qquad \left(\left|\frac{z-1}{z+1}\right|<1\right) \tag{5}$$

using

$$\Gamma(\lambda+1-k) = (-1)^{-k}\frac{\Gamma(\lambda+1)}{[-\lambda]_k} \tag{35}$$

and hence

$$\Gamma(\delta-k) = (-1)^{-k}\frac{\Gamma(\delta)}{[1-\delta]_k} \ . \tag{36}$$

(ii) $$\varphi_{[6](z,\nu)} = (z-1)^{1/2}\left((z+1)^{-\nu}\cdot(z-1)^{-(\nu+1)}\right)_{-(\nu+1/2)} \tag{37}$$

$$= (z-1)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(1-\nu-\frac{1}{2})}{k!\,\Gamma(1-\nu-\frac{1}{2}-k)}((z+1)^{-\nu})_{-(\nu+1/2)-k}((z-1)^{-(\nu+1)})_k \tag{38}$$

(by Lemma (iv)

$$= i\,e^{i\pi\nu}\frac{\Gamma(-\frac{1}{2})}{\Gamma(\nu)}(z+1)^{1/2}(z-1)^{-(\nu+1/2)}\sum_{k=0}^{\infty}\frac{[\frac{1}{2}+\nu]_k\,[\nu+1]_k}{k!\,[\frac{3}{2}]_k}\left(\frac{z+1}{z-1}\right)^k \tag{39}$$

(by Lemma (i))

$$= -ie^{i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(\nu)}(z-1)^{-\nu}\left(\tfrac{z+1}{z-1}\right)^{1/2}\ {}_2F_1(\nu+\tfrac{1}{2},\nu+1;\tfrac{3}{2};\tfrac{z+1}{z-1}) \qquad \left(\left|\frac{z+1}{z-1}\right|<1\right). \tag{6}$$

(iii)
$$\varphi_{[7](z,\nu)} = (z-1)^{1/2}\left((z-1)^{\nu-1}\cdot(z+1)^{\nu}\right)_{\nu-1/2} = \varphi_{[5](z,-\nu)} \tag{40}$$

$$= ie^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1-\nu)}(z+1)^{\nu}\ {}_2F_1(\tfrac{1}{2}-\nu,-\nu;\tfrac{1}{2};\tfrac{z-1}{z+1}) \qquad \left(\left|\frac{z-1}{z+1}\right|<1\right) \tag{7}$$

(iv)
$$\varphi_{[8](z,\nu)} = (z-1)^{1/2}\cdot\left((z+1)^{\nu}(z-1)^{\nu-1}\right)_{\nu-1/2} = \varphi_{[6](z,-\nu)} \tag{41}$$

$$= -ie^{-i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(-\nu)}(z-1)^{\nu}\left(\tfrac{z+1}{z-1}\right)^{1/2}\ {}_2F_1(\tfrac{1}{2}-\nu,1-\nu;\tfrac{3}{2};\tfrac{z+1}{z-1}) \qquad \left(\left|\frac{z+1}{z-1}\right|<1\right) \tag{8}$$

Proof of Group I V ;

We have

(i)
$$\varphi_{[9](z,\nu)} = (z+1)^{1/2}\left((z-1)^{-\nu}\cdot(z+1)^{-(\nu+1)}\right)_{-(\nu+1/2)} \tag{42}$$

$$= (z+1)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(1-\nu-\frac{1}{2})}{k!\,\Gamma(1-\nu-\frac{1}{2}-k)}((z-1)^{-\nu})_{-(\nu+1/2)-k}((z+1)^{-(\nu+1)})_k \tag{43}$$

$$= e^{i\pi(\nu+1/2)}\frac{\Gamma(-\frac{1}{2})}{\Gamma(\nu)}(z+1)^{-\nu}\left(\frac{z-1}{z+1}\right)^{1/2}\sum_{k=0}^{\infty}\frac{[\frac{1}{2}+\nu]_k\,[\nu+1]_k}{k!\,[\frac{3}{2}]_k}\left(\frac{z-1}{z+1}\right)^k \tag{44}$$

$$= -ie^{i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(\nu)}(z+1)^{-\nu}\left(\tfrac{z-1}{z+1}\right)^{1/2}\ {}_2F_1(\nu+\tfrac{1}{2},\nu+1;\tfrac{3}{2};\tfrac{z-1}{z+1}) \qquad \left(\left|\frac{z-1}{z+1}\right|<1\right) \tag{9}$$

(ii)
$$\varphi_{[10](z,\nu)} = (z+1)^{1/2}\left((z+1)^{-(\nu+1)}\cdot(z-1)^{-\nu}\right)_{-(\nu+1/2)} \tag{45}$$

$$= (z+1)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(1-\nu-\frac{1}{2})}{k!\,\Gamma(1-\nu-\frac{1}{2}-k)}((z+1)^{-(\nu+1)})_{-(\nu+1/2)-k}((z-1)^{-\nu})_k \tag{46}$$

$$= ie^{i\pi\nu}\frac{\Gamma(\frac{1}{2})}{\Gamma(1+\nu)}(z-1)^{-\nu}\sum_{k=0}^{\infty}\frac{[\frac{1}{2}+\nu]_k\,[\nu]_k}{k!\,[\frac{1}{2}]_k}\left(\frac{z+1}{z-1}\right)^k \tag{47}$$

$$= ie^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z-1)^{-\nu}\ {}_2F_1(\nu+\tfrac{1}{2},\nu;\tfrac{1}{2};\tfrac{z+1}{z-1}) \qquad \left(\left|\frac{z+1}{z-1}\right|<1\right). \tag{10}$$

(iii) $$\varphi_{[11](z,\nu)} = (z+1)^{1/2}\left((z-1)^{\nu}\cdot(z+1)^{\nu-1}\right)_{\nu-1/2} = \varphi_{[9](z,-\nu)} \tag{48}$$

$$= -ie^{-i\pi\nu}\frac{2\sqrt{\pi}}{\Gamma(-\nu)}(z+1)^{\nu}\left(\tfrac{z-1}{z+1}\right)^{1/2}{}_2F_1\left(\tfrac{1}{2}-\nu,\ 1-\nu\,;\tfrac{3}{2}\,;\tfrac{z-1}{z+1}\right) \quad \left(\left|\frac{z-1}{z+1}\right|<1\right) \tag{11}$$

(iv) $$\varphi_{[12](z,\nu)} = (z+1)^{1/2}\cdot\left((z+1)^{\nu-1}(z-1)^{\nu}\right)_{\nu-1/2} = \varphi_{[10](z,-\nu)} \tag{49}$$

$$= ie^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1-\nu)}(z-1)^{\nu}\ {}_2F_1\left(\tfrac{1}{2}-\nu,\ -\nu\,;\tfrac{1}{2}\,;\tfrac{z+1}{z-1}\right) \quad \left(\left|\frac{z+1}{z-1}\right|<1\right). \tag{12}$$

§ 4. Some Examples

[I] Example for Homogeneous Equation

(i) When $\nu = -1$, we have

$$\varphi_2\cdot(z^2-1)+\varphi_1\cdot z-\varphi=0 \tag{1}$$

and

$$\varphi=\varphi_{[1](z,-1)}=(z^2-1)^{1/2} \tag{2}$$

from § 2. (1) and § 2. (2), respectively.

The function shown by (2) satisfies equation (1) clearly.

(ii) When $\nu = 2$, we have

$$\varphi_2\cdot(z^2-1)+\varphi_1\cdot z-\varphi\cdot 4=0 \tag{3}$$

and

$$\varphi=\varphi_{[2](z,2)}=((z^2-1)^{3/2})_1=(z^2-1)^{1/2}3z \tag{4}$$

from § 2. (1) and § 2. (3), respectively.

The function shown by (4) satisfies equation (3) clearly.

(iii) When $\nu = 1/2$, we have

$$\varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot 1/4 = 0 \tag{5}$$

and

$$\varphi = \varphi_{[7](z,1/2)} = (z-1)^{1/2}((z-1)^{-1/2} \cdot (z+1)^{1/2})_0 = (z+1)^{1/2} \tag{6}$$

from § 2. (1) and § 2. (8), respectively.

The function shown by (6) satisfies equation (5) clearly.

(iv) When $\nu = 1/2$, we have (5) from § 2. (1).

And we have

$$\varphi = \varphi_{[1](z,1/2)} = ((z^2-1)^{-1})_{-3/2} \tag{7}$$

$$= -iz^{-1/2} \sum_{k=0}^{\infty} \frac{[\,1\,]_k \Gamma(2k+1/2)}{k!\,\Gamma(2k+2)} \left(\frac{1}{z^2}\right)^k \quad (|1/z^2| < 1), \tag{8}$$

$$\varphi_1 = ((z^2-1)^{-1})_{-1/2} = iz^{-3/2} \sum_{k=0}^{\infty} \frac{[\,1\,]_k \Gamma(2k+3/2)}{k!\,\Gamma(2k+2)} \left(\frac{1}{z^2}\right)^k \quad (|1/z^2| < 1), \tag{9}$$

$$\varphi_2 = ((z^2-1)^{-1})_{1/2} = -iz^{-5/2} \sum_{k=0}^{\infty} \frac{[\,1\,]_k \Gamma(2k+5/2)}{k!\,\Gamma(2k+2)} \left(\frac{1}{z^2}\right)^k \quad (|1/z^2| < 1), \tag{10}$$

by Theorem E (i), respectively.

Therefore, applyibg (8), (9) and (10) we obtain

$$\text{LHS of (5)} = -iz^{-1/2} \sum_{k=0}^{\infty} S(k)\Gamma(2k+5/2)\, T^k + iz^{-5/2} \sum_{k=0}^{\infty} S(k)\Gamma(2k+5/2)\, T^k$$

$$+ iz^{-1/2} \sum_{k=0}^{\infty} S(k)\Gamma(2k+3/2)\, T^k + i(1/4)z^{-1/2} \sum_{k=0}^{\infty} S(k)\Gamma(2k+1/2)\, T^k \tag{11}$$

$$(S(k) = [\,1\,]_k / k!\,\Gamma(2k+2), \quad T = 1/z^2, \quad |1/z^2| < 1)$$

$$= iz^{-1/2}(-\Gamma(5/2) + \Gamma(3/2) + (1/4)\Gamma(1/2))$$

$$+ iz^{-1/2-2}\left(-\frac{\Gamma(2+5/2)}{\Gamma(4)} + \Gamma(5/2) + \frac{\Gamma(2+3/2)}{\Gamma(4)} + \frac{\Gamma(2+1/2)}{4\Gamma(4)}\right)$$

$$+ iz^{-1/2-4}\left(-\frac{\Gamma(4+5/2)}{\Gamma(6)} + \frac{\Gamma(2+5/2)}{\Gamma(4)} + \frac{\Gamma(4+3/2)}{\Gamma(6)} + \frac{\Gamma(4+1/2)}{4\Gamma(6)}\right)$$

$$+iz^{-1/2-6}(\cdots\cdots\cdots)+iz^{-1/2-8}(\cdots\cdots\cdots)+\cdots\cdots\cdots \tag{12}$$

$$=iz^{-1/2}\left(-\frac{3}{4}+\frac{3}{4}\right)\Gamma(1/2)+iz^{-5/2}\left(-\frac{105}{6\cdot 4^2}+\frac{3}{4}+\frac{15}{6\cdot 8}+\frac{3}{6\cdot 4^2}\right)\Gamma(1/2)$$

$$+iz^{-9/2}\left(-\frac{11\cdot 9\cdot 7\cdot 5\cdot 3}{120\cdot 4^3}+\frac{105}{6\cdot 4^2}+\frac{9\cdot 105}{120\cdot 4^2\cdot 2}+\frac{105}{120\cdot 4^3}\right)\Gamma(1/2)$$

$$+iz^{-13/2}(\cdots\cdots\cdots)\Gamma(1/2)+\cdots\cdots\cdots\cdots \tag{13}$$

$$=iz^{-1/2}\cdot 0+iz^{-5/2}\cdot 0+iz^{-9/2}\cdot 0+\cdots\cdots\cdots=0\,. \tag{14}$$

§ 5. Representations for The Solutions (in § 3) With Use of Socalled Chebyshev's Functions

[I] The Chebyshev's functions are defined as

$$T(z,\nu)={}_2F_1(-\nu,\,\nu;\,\tfrac{1}{2}\,;\,\tfrac{1-z}{2}\,),\quad \left(\left|\tfrac{1-z}{2}\right|<1\right)\,. \tag{1}$$

(First kind Chebyshev's function)

and

$$U(z,\nu)=\nu(1-z^2)^{1/2}\;{}_2F_1(1-\nu,\,1+\nu\,;\,\tfrac{3}{2}\,;\,\tfrac{1-z}{2}\,),\quad \left(\left|\tfrac{1-z}{2}\right|<1\right) \tag{2}$$

(Second kind Chebyshev's function)

respectively.

Here we have the identity

$${}_2F_1(\alpha,\,\beta\,;\gamma\;;z\,)=(1-z)^{-\beta}\;{}_2F_1(\gamma-\alpha,\,\beta\,;\gamma\;;\,\tfrac{z}{z-1}\,), \tag{3}$$

$$(\,|\,z\,|<1,\;\left|\tfrac{z}{z-1}\right|<1,\;\gamma\notin\mathbf{Z}_0^-\,)$$

hence

$${}_2F_1(-\nu,\,\nu\,;\tfrac{1}{2}\,;\,\tfrac{1-z}{2}\,)=\left(\tfrac{1+z}{2}\right)^{-\nu}\;{}_2F_1(\tfrac{1}{2}+\nu\,,\,\nu\,;\,\tfrac{1}{2}\,;\,\tfrac{z-1}{z+1}\,), \tag{4}$$

$$(\,\left|\tfrac{1-z}{2}\right|<1,\;\left|\tfrac{z-1}{z+1}\right|<1\,)$$

and

$$ {}_2F_1(1+\nu,\ 1-\nu\,;\ \tfrac{3}{2}\ ;\ \tfrac{1-z}{2}\,)=\left(\tfrac{1+z}{2}\right)^{\nu-\nu}\ {}_2F_1(1-\nu,\ \tfrac{1}{2}-\nu\ ;\ \tfrac{3}{2}\ ;\ \tfrac{z-1}{z+1}), \qquad (5)$$

$$\left(\left|\tfrac{1-z}{2}\right|<1,\quad \left|\tfrac{z-1}{z+1}\right|<1\right).$$

Therefore, we obtain

$$T(z,\nu)=\left(\tfrac{1+z}{2}\right)^{-\nu}\ {}_2F_1(\tfrac{1}{2}+\nu,\ \nu\ ;\ \tfrac{1}{2}\ ;\ \tfrac{z-1}{z+1}), \qquad (6)$$

and

$$U(z,\nu)=\frac{\nu}{2^{\nu-1}}(1+z)^{\nu}\left(\tfrac{1-z}{1+z}\right)^{1/2}\ {}_2F_1(1-\nu,\ \tfrac{1}{2}-\nu\ ;\ \tfrac{3}{2}\ ;\ \tfrac{z-1}{z+1})\ . \qquad (7)$$

We have then

(i) $$\varphi_{[5](z,\nu)}=i\,e^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z+1)^{-\nu}\ {}_2F_1(\tfrac{1}{2}+\nu,\ \nu\ ;\ \tfrac{1}{2}\ ;\ \tfrac{z-1}{z+1}\,)$$

$$=i\,e^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}2^{\nu}T(z,\nu),\qquad \left(\left|\tfrac{z-1}{z+1}\right|<1\right) \qquad (8)$$

(ii) $$\varphi_{[7](z,\nu)}=\varphi_{[5](z,-\nu)}=i\,e^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1-\nu)}2^{-\nu}\,T(z,-\nu)\ , \qquad (9)$$

(iii) $$\varphi_{[11](z,\nu)}=-\,i2\,e^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(-\nu)}(z+1)^{\nu}\left(\tfrac{z-1}{z+1}\right)^{1/2}{}_2F_1(\tfrac{1}{2}-\nu,\ 1-\nu\ ;\ \tfrac{3}{2}\ ;\ \tfrac{z-1}{z+1})$$

$$=-\,ie^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(-\nu)}\cdot\frac{2^{\nu}}{\nu}\ U(z,\nu),\qquad \left(\left|\tfrac{z-1}{z+1}\right|<1\right) \qquad (10)$$

and

(iv) $$\varphi_{[9](z,\nu)}=\varphi_{[11](z,-\nu)}=-\,ie^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(\nu)}\cdot\frac{2^{-\nu}}{(-\nu)}U(z,-\nu)\ , \qquad (11)$$

respectively, using (6) and (7).

Next we have

$$T(-z,\nu)=e^{-i\pi\nu}2^{\nu}(z-1)^{-\nu}\ {}_2F_1(\tfrac{1}{2}+\nu,\ \nu\ ;\ \tfrac{1}{2}\ ;\ \tfrac{z+1}{z-1}\,), \qquad (12)$$

and

$$U(-z,\nu)=e^{i\pi\nu}\frac{\nu}{2^{\nu-1}}(z-1)^{\nu}\left(\tfrac{1+z}{1-z}\right)^{1/2}\ {}_2F_1(1-\nu,\ \tfrac{1}{2}-\nu\ ;\ \tfrac{3}{2}\ ;\ \tfrac{z+1}{z-1})\ . \qquad (13)$$

setting $-z$ instead of z in (6) and (7), respectively.

Therefore, we obtain

(v) $$\varphi_{[10](z,\nu)} = i\,e^{i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}(z-1)^{-\nu}\,{}_2F_1\!\left(\tfrac{1}{2}+\nu,\ \nu\,;\ \tfrac{1}{2}\,;\ \tfrac{z+1}{z-1}\right)$$

$$= i\,2^{-\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}T(-z,\nu), \qquad \left(\left|\tfrac{z+1}{z-1}\right|<1\right) \tag{14}$$

(vi) $$\varphi_{[12](z,\nu)} = \varphi_{[10](z,-\nu)} = i\,2^{\nu}\frac{\sqrt{\pi}}{\Gamma(1-\nu)}T(-z,-\nu)\ , \tag{15}$$

(vii) $$\varphi_{[8](z,\nu)} = -\,i2\,e^{-i\pi\nu}\frac{\sqrt{\pi}}{\Gamma(-\nu)}(z-1)^{\nu}\left(\tfrac{z+1}{z-1}\right)^{1/2}{}_2F_1\!\left(\tfrac{1}{2}-\nu,\ 1-\nu\,;\ \tfrac{3}{2}\,;\ \tfrac{z+1}{z-1}\right)$$

$$= i\frac{\sqrt{\pi}}{\Gamma(1-\nu)}2^{\nu}\,U(-z,\nu), \qquad \left(\left|\tfrac{z+1}{z-1}\right|<1\right) \tag{16}$$

and

(viii) $$\varphi_{[6](z,\nu)} = \varphi_{[8](z,-\nu)} = i\,2^{-\nu}\frac{\sqrt{\pi}}{\Gamma(1+\nu)}U(-z,-\nu)\ , \tag{17}$$

respectively, using (12) and (13).

[II] Solutions of Group I and II (whose index $\nu = n \in \boldsymbol{Z}$) in § 2.

The polynomials of Chebyshev are defined as

$$T(z,n) = \cos(n\arccos z) = \frac{(-1)^n}{(2n-1)!!}\sqrt{1-z^2}\ \frac{d^n}{dz^n}(1-z^2)^{n-1/2} \tag{18}$$

(First kind Chebyshev's polynomials)

abd

$$U(z,n) = \sin(n\arccos z) = \frac{n(-1)^{n-1}}{(2n-1)!!}\cdot\frac{d^{n-1}}{dz^{n-1}}(1-z^2)^{n-1/2} \tag{19}$$

(Second kind Chebyshev's polynomials)

where $n \in \boldsymbol{Z}^+$ and

$$(2n-1)!! = (2n-1)(2n-3)\cdot\cdots\cdot 3\cdot 1\ . \tag{20}$$

1) Now we have

$$\varphi_{[2](z,n)} = \left((z^2-1)^{n-1/2}\right)_{n-1} = -i(-1)^n\left((1-z^2)^{n-1/2}\right)_{n-1} \quad (n\in\boldsymbol{Z}) \tag{21}$$

$$= \begin{cases} -i(-1)^n(d/dz)^{n-1}(1-z^2)^{n-1/2}, & (n\in\boldsymbol{Z}^+) \quad (22) \\ -i(-1)^{-n}\int\cdots\int(1-z^2)^{-(n+1/2)}(dz)^{n+1}, & (n\in\boldsymbol{Z}_0^+) \quad (23) \end{cases}$$

Therefore, we obtain

$$\varphi_{[2](z,n)} = i\frac{(2n-1)!!}{n}U(z,n) = \varphi_{[1](z,-n)} \quad (n \in \mathbf{Z}^+) \ , \tag{24}$$

from (22) and (19).

2) next we have

$$\varphi_{[4](z,n)} = (z^2-1)^{1/2}\left((z^2-1)^{n-1/2}\right)_{n1} = (-1)^n(1-z^2)^{1/2}\left((1-z^2)^{n-1/2}\right)_n \quad (n \in \mathbf{Z}) \tag{25}$$

$$= \begin{cases} (-1)^n(1-z^2)^{1/2}(d/dz)^n(1-z^2)^{n-1/2}, & (n \in \mathbf{Z}_0^+) \quad (26) \\ (-1)^{-n}(1-z^2)^{1/2}\int\cdots\int(1-z^2)^{-(n+1/2)}(dz)^n, & (n \in \mathbf{Z}^+) \quad (27) \end{cases}$$

Therefore, we obtain

$$\varphi_{[4](z,n)} = (2n-1)!!\ T(z,n) = \varphi_{[3](z,-n)} \quad (n \in \mathbf{Z}^+) \ , \tag{28}$$

from (26) and (19).

Hitherto the solutions to the homogeneous Chebyshev's equation are shown by the differential forms only as (18) and (19)..Then the representations of the integral forms like as (23) and (27) are fresh.

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